INSTRUCTOR'S MANUAL FOR

FUNDAMENTALS OF QUEUEING THEORY

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FOURTH EDITION

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CHAPTER 1

Introduction

1.1

Prob.	Calling Units	Service Function	Discipline	Capacity	No. Servers	No. Stages
(a)	Airplanes	Landing runways	FCFS (PRI in emergency)	Stack ($\approx \infty$)	No. runways	1-landing only; 2-landing and taxiing
(b)	Filled Grocery carts	Checker(and bagger)	FCFS (with jockeying)	($\approx \infty$)	With jockeying and channel choice acts like a c-server model	1
(c)	People	Clerks	same as (b)	same as (b)	same as (b)	1
(d)	Cars	Paying toll (toll booth)	FCFS	∞	1 or more (in fog, acts like indep. single channels no choice or jockeying)	1
(e)	Cars	Gas filling	FCFS	Finite	No. of pump islands (similar to (b) although jockeying difficult)	1

(f)	Cars	Car-wash building	FCFS	Finite	Generally 1	Many, with no storage between stages
(g)	Calls	Lines in switchboard	FCFS	Finite	No. of lines	1
(h)	Patients	Doctor (could be batch service)	Fixed as to appoint-ments	Finite seating capacity and waiting room	1, unless a clinic	Usually 1 but could be several
(i)	Tourists	Tour group	FCFS	($\approx \infty$)	1 or more	Multiple
(j)	Components	Operations and inspection batch service	FCFS	Finite	1 or more	4
(k)	Programs	Processing Programs	FCFS (or PRI)	same as (b)	1	1

1.2 One could give a variety of illustrations, e.g., people calling into a bank to find their account status. The customers are the calls, it is generally a multi-stage process, where first an automated message of which button to press depending on what's desired is received, and then, after pressing the appropriate button, getting the desired information automatically or asking for a customer representative. We would have finite capacity - if all lines are tied up, a busy signal results and the call must be replaced. It is multi-stage and would usually be a multi-server queue, with a FCFS discipline. Another example might be a bakery, where upon entering, the customer takes a number, so that we have a true, FCFS, multi-server queue with a single waiting line (the queue being the numbers). It would be a single-stage process, since a given server serves only one customer at a time. The capacity would be finite, although there is usually enough space so that it is essentially infinite. As a final example, consider a blood donor center. We have a multi-stage process (check-in, filling out information, blood pressure and clotting-time checks, and finally giving the blood). Some stages have a single server and others have multiple servers. It is generally an appointment system, but if it is a drop-in center, customers can arrive completely randomly and we would have a FCFS discipline. There is a finite capacity in that if the waiting room is completely filled, donors might be asked to come back at another time.

1.3 The parameters are $\lambda = 40/\text{h}$ and $1/\mu = 5.5\,\text{min}$. Using units of hours, $\mu = 60/5.5 \doteq 10.91/\text{h}$. The utilization should be less than 1, so $\lambda/c\mu \doteq 40/(10.91c)$, which implies that $c > 40/10.91 \doteq 3.67$. At least 4 are required to achieve steady state.

1.4 $\text{Lq} = \lambda \text{Wq} = (3/\text{min})([75/60]\,\text{min}) = 3.75$ or, say, 4. The 3.75 number is, of course, the average number in the queue. We may wish to provide 5 or 6 slots to guarantee that most callers get into the queue.

1.5 (a): From Table 1.2, probability of any server busy, $p_b = 1 - .01 = .99$. Now $p_b = \lambda/c\mu = r/c$, so that $r = c \times p_b = 2 \times .99 = 1.98$. With 3 servers, $p_b = 1.98/3 = .66$, so that now, each server is idle 34% of the time, more than enough time for breaks. (b): If now, μ becomes $0.8 \times \mu$, $p_b = \lambda/(3 \times 0.8 \times \mu) = (1/0.8)(r/3) = 1.98/(2.4) = 0.825$. This still gives an idle percentage for each server of 17.5%, again more than enough time for breaks. (c): Let μ' =the new service rate, so that $1/\mu' = .8(1/\mu)$, hence $\mu' = 1.25\mu$. Thus $p_b = \lambda/(2\mu') = \lambda/(2 \times 1.25\mu) = (1/1.25)(.99) = 0.792$, or an idle percent per server of 20.8%, a cheaper solution giving each server enough time for breaks.

1.6 Let T be the total waiting time. If, when you arrive, the person in service is just about finished, then you wait on average eight service times (yours and the seven ahead of you) or $\mathrm{E}[T] = 8(2.5\,\mathrm{min}) = 20\,\mathrm{min}$. If, when you arrive, the person in service is just beginning, then you wait on average nine service times or $\mathrm{E}[T] = 9(2.5\,\mathrm{min}) = 22.5\,\mathrm{min}$. The average wait is somewhere in between.

Assuming the latter case, T is the sum of 9 IID normal random variables each with mean 2.5 and standard deviation 0.5. So T is a normal random variable with mean 22.5 and standard deviation $\sqrt{(9 \cdot 0.5^2)} = 1.5$. Then $\Pr\{T > 30\,\mathrm{min}\} = \Pr\{Z > (30 - 22.5)/1.5\} = \Pr\{Z > 5\}$, where Z is a standard normal random variable. From standard normal tables, $\Pr\{\mathrm{Z} > 5\} \doteq 0$.

1.7
- **(a)** Apply Little's law to the system of active players in the league. The average number of active players in the league is represented by L, where $L = 32 \cdot 67 = 2,144$. The average rate that players enter the league is represented by λ, where $\lambda = 32 \cdot 7 = 224$ per year. The average time spent in the league is represented by W. By Little's law, $W = L/\lambda = 2144/224 = 9.57$ years.
- **(b)** Here, it is given that $W = 3.5$ years. As before $L = 2,144$ (the number of active players in the league). The average rate that players enter the league is $\lambda = L/W = 2,144/3.5 \approx 613$ per year. Since 224 players are drafted each year, an average of $613 - 224 = 389$ players enter the league without being drafted. (This analysis assumes that a player who leaves the league never returns.)

1.8 Consider the university as a system where students enter by enrolling at the university. The average undergraduate enrollment is an estimate for L (so $L = 16,800$). The average number of new students per year (the sum of the middle two columns) is an estimate for λ (so $\lambda = 4,052$ per year). W is an estimate for the average time an undergraduate spends at the university. By Little's formula, $W = L/\lambda \approx 4.1$ years. (The main assumption here is that the system is operating in steady-state. This may not be a valid assumption, for example, if enrollment were growing. However, this particular example does not indicate a noticeable growth trend.)

1.9 Apply Little's law to the set of homes on the market. The average number of homes on the market is estimated as $L = 50$. The rate that homes enter the market is estimated as $\lambda = 5$ per week. By Little's law, a home is on the market for an average of $W = L/\lambda = 10$ weeks before it is sold. This assumes that the observed numbers are representative of long-term averages. Furthermore, it is assumed that you have no additional information that might change your estimate. For example, if you price your home at a very low price, you will probably sell it more quickly than the average.

1.10 We use the Delay Analysis for Sample Single-Server Queue model in the Basic Model category in QtsPlus:

DELAY ANALYSIS FOR SAMPLE SINGLE-SERVER QUEUE

This is a basic line waiting-time analysis for a sample G/G/1 queue constructed from an input sequence of interarrival and service times.

Clear Old Data

Put data below into two columns of equal length.
Enter data and then press "Solve" button.

Solve

Output:

Number of Observations	20
Total time horizon	147
Mean interarrival time	7.35
Arrival rate (λ)	0.136054422
Mean service time	6.2
Service rate (μ)	0.161290323
Empirical traffic intensity (ρ)	84.35%
Average line delay (Wq)	3.95
Average system wait (W)	10.15

Customer n	**Line Delays Wq(n)**	**System Waits W(n)**	**Service Time S(n)**	**Inter-arrival Time T(n)**
0	***N/A***	***N/A***	***N/A***	1.
1	0.0	3.0	3.	9.
2	0.0	7.0	7.	6.
3	1.0	10.0	9.	4.
4	6.0	15.0	9.	7.
5	8.0	18.0	10.	9.
6	9.0	13.0	4.	5.
7	8.0	16.0	8.	8.
8	8.0	13.0	5.	4.
9	9.0	14.0	5.	10.
10	4.0	7.0	3.	6.
11	1.0	7.0	6.	12.
12	0.0	3.0	3.	6.
13	0.0	5.0	5.	8.
14	0.0	4.0	4.	9.
15	0.0	9.0	9.	5.
16	4.0	13.0	9.	7.
17	6.0	14.0	8.	8.
18	6.0	12.0	6.	8.
19	4.0	12.0	8.	7.
20	5.0	8.0	3.	

1.11 Using QtsPlus Delay Analysis for Sample Single-Server Queue model in the Basic Model category:

DELAY ANALYSIS FOR SAMPLE SINGLE-SERVER QUEUE

This is a basic line waiting-time analysis for a sample G/G/1 queue constructed from an input sequence of interarrival and service times.

Output:

Number of Observations	10
Total time horizon	60
Mean interarrival time	6
Arrival rate (λ)	0.166666667
Mean service time	4.6
Service rate (μ)	0.217391304
Empirical traffic intensity (ρ)	76.67%
Average line delay (Wq)	1.7
Average system wait (W)	6.3

Clear Old Data

Put data below into two columns of equal length.
Enter data and then press "Solve" button.

Solve

Customer n	Line Delays Wq(n)	System Waits W(n)	Service Time S(n)	Inter-arrival Time T(n)
0	*N/A*	*N/A*	*N/A*	5.
1	0.0	2.0	2.	5.
2	0.0	7.0	7.	5.
3	2.0	8.0	6.	5.
4	3.0	9.0	6.	5.
5	4.0	10.0	6.	5.
6	5.0	8.0	3.	5.
7	3.0	4.0	1.	5.
8	0.0	4.0	4.	5.
9	0.0	1.0	1.	5.
10	0.0	10.0	10.	

1.12 The following table lists various statistics associated with each customer. "# in System" and "# in Queue" refer to the number of customers in the system and queue as seen by the arriving customer.

Customer # / Arrival Time	Service Start Time	Exit Time	Time in Queue	# in System	# in Queue
1	1.00	3.22	0.00	0	0
2	3.22	4.98	1.22	1	0
3	4.98	7.11	1.98	2	1
4	7.11	7.25	3.11	2	1
5	7.25	8.01	2.25	2	1
6	8.01	8.71	2.01	3	2
7	8.71	9.18	1.71	4	3
8	9.18	9.40	1.18	3	2
9	9.40	9.58	0.40	2	1
10	10.00	12.41	0.00	0	0
11	12.41	12.82	1.41	1	0
12	12.82	13.28	0.82	2	1
13	13.28	14.65	0.28	1	0
14	14.65	14.92	0.65	1	0
15	15.00	15.27	0.00	0	0

The values in the table are computed as follows:

- The exit time is the service-start time plus the service duration.
- The service-start time is the maximum of the exit time of the previous customer and the arrival time of the customer in question. (The first customer starts service immediately upon arrival.)
- The time in queue is the service-start time minus the arrival time.
- The number in system is the number of previously arriving customers whose exit time is after the arrival time of the customer in question.
- The number in queue is the number in system minus one, with a minimum value of zero.

$L_q^{(A)}$ is the average of the last column. $L_q^{(A)} = 12/15 = 0.8$. L_q is the total person minutes spent in the queue (the sum of the "Time in Queue" column) divided by the total time interval. $L_q = 17.02/15.27 = 1.1146$. Note that $L_q \neq L_q^{(A)}$.

1.13 From differential equation theory, the solution to $\frac{dy(t)}{dt} + \phi(t)y(t) = \psi(t)$ is $y(t) = ce^{-\int \phi(t)dt} + e^{-\int \phi(t)dt} \int e^{-\int \phi(t)dt \psi(t)dt}$. So $\frac{dp_0(t)}{dt} + \lambda p_0(t) = 0; p_0(0) = 1$.

Set $\phi(t) = \lambda$ and $\psi(t) = 0 \Rightarrow p_0(t) = ce^{-\lambda t}$.

From the boundary condition: $1 = ce^0 \Rightarrow c = 1$. Therefore, $p_0(t) = e^{-\lambda t}$.

$\frac{dp_1(t)}{dt} + \lambda p_1(t) = \lambda p_0(t) = \lambda e^{-\lambda t}; p_1(0) = 0$.

Set $\phi(t) = \lambda$ and $\psi(t) = \lambda e^{-\lambda t} \Rightarrow p_1(t) = ce^{-\lambda t} + \lambda te^{-\lambda t}$.

From the boundary condition: $0 = ce^0 + 0 \Rightarrow c = 0 \Rightarrow p_1(t) = \lambda te^{-\lambda t}$.

$\frac{dp_2(t)}{dt} + \lambda p_2(t) = \lambda p_1(t) = \lambda^2 te^{-\lambda t}; p_2(0) = 0$.

Set $\phi(t) = \lambda$ and $\psi(t) = \lambda^2 te^{-\lambda t} \Rightarrow p_2(t) = ce^{-\lambda t} + \frac{(\lambda t)^2}{2} e^{-\lambda t}$.

From the boundary condition: $c = 0 \Rightarrow p_2(t) = \frac{(\lambda t)^2}{2} e^{-\lambda t}$.

$\frac{dp_3(t)}{dt} + \lambda p_3(t) = \lambda p_2(t) = \frac{\lambda^3 t^2}{2} e^{-\lambda t}; p_3(0) = 0$.

Set $\phi(t) = \lambda$ and $\psi(t) = \frac{\lambda^3 t^2}{2} e^{-\lambda t} \Rightarrow p_3(t) = ce^{-\lambda t} + \frac{(\lambda t)^3}{3 \cdot 2} e^{-\lambda t}$.

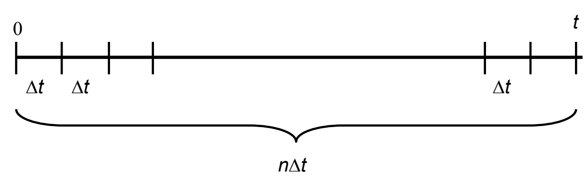
The boundary condition gives $c = 0 \Rightarrow p_3(t) = \frac{(\lambda t)^3}{3!} e^{-\lambda t}$.

Now assume $p_{n-1}(t) = \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t}$. Set $\phi(t) = \lambda$ and $\psi(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t} \Rightarrow p_n(t) = ce^{-\lambda t} + \frac{(\lambda t)^n}{n!} e^{-\lambda t}$ and boundary condition gives $c = 0 \Rightarrow p_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$.

1.14

$$
\begin{aligned}
p_n(t) &= \frac{\tau^n e^{-\tau}}{n!}, \tau = \lambda t, n = 0, 1, 2 \ldots \\
M_{N(t)}(\theta) &= E[e^{\theta N(t)}] = \sum_{n=0}^{\infty} \frac{\tau^n e^{-\tau} e^{\theta n}}{n!} = e^{-\tau} \sum_{n=0}^{\infty} \frac{(\tau e^{\theta})^n}{n!} = e^{-\tau} e^{\tau e^{\theta}} = e^{\tau(e^{\theta}-1)} \\
E[N(t)] &= \left. \frac{dM_{N(t)}(\theta)}{d\theta} \right|_{\theta=0} = \left. \tau e^{\theta} e^{\tau(e^{\theta}-1)} \right|_{\theta=0} = \tau \\
E[(N(t) - E[N(t)])^2] &= E[(N(t))^2] - \{E[N(t)]\}^2 = \left. \frac{d^2 M_{N(t)}(\theta)}{d\theta^2} \right|_{\theta=0} - \tau^2 \\
&= [\tau e^{\theta} e^{\tau(e^{\theta}-1)} + \tau^2 e^{2\theta} e^{\tau(e^{\theta}-1)}]_{\theta=0} - \tau^2 = \tau + \tau^2 - \tau^2 = \tau
\end{aligned}
$$

1.15



Divide the interval $[0, t]$ into n subintervals of length Δt, so that $t = n\,\Delta t$. The probability of one arrival in a subinterval is

$$
p \equiv \Pr\{\text{one arrival in } \Delta t\} = \lambda \Delta t + o(\Delta t) \approx \frac{\lambda t}{n}.
$$

The probability of more than one arrival in a subinterval is $o(\Delta t)$, which can be made arbitrarily small. Assuming that there can be at most one arrival in a subinterval and using the assumption of independence of non-overlapping intervals, the total number of arrivals in $[0, t]$ is the sum of n Bernoulli trials. This follows a binomial distribution:

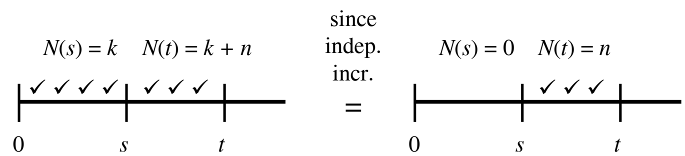
$$
\begin{aligned}
b(x; n, p) &= \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n \\
&= \frac{n(n-1)\cdots(n-x+1)}{x!} p^x (1-p)^n (1-p)^{-x} \\
&= \frac{1 \cdot (1 - 1/n) \cdots (1 - \frac{x-1}{n})}{x!} (np)^x \left(1 - \frac{\lambda t}{n}\right)^n \left(1 - \frac{\lambda t}{n}\right)^{-x}.
\end{aligned}
$$

So,

$$
\lim_{n \to \infty} b(x; n, p) = \frac{1}{x!} (\lambda t)^x e^{-\lambda t}, \quad x = 0, 1, \ldots,
$$

which is the Poisson distribution.

1.16 Consider



Let $P_r\{N(t) - N(s) = n\} = q_n(t, s)$.

So $q_n(t + \Delta t, s) = q_n(t, s)[1 - \lambda\Delta t] + q_{n-1}(t, s)\lambda\Delta t + o(\Delta t) \quad n > 0$.

$q_0(t + \Delta t, s) = q_0(t, s)[1 - \lambda\Delta t] + o(\Delta t)$.

Rearranging & dividing by Δt, then taking $\lim \Delta t \to 0$ gives

$$\frac{\partial q_n(t,s)}{\partial t} = -\lambda q_n(t,s) + \lambda q_{n-1}(t,s)$$
$$\frac{\partial q_0(t,s)}{\partial t} = -\lambda q_0(t,s)$$

Solve in a similar manner to (1.11) & (1.12) by the general solution to a first order linear differential equation was in Problem 1.13 solution. Here, however, the boundary conditions are

$$q_0(s,s) = 1, q_n(s,s) = 0, n \neq 0.$$
$$q_0(t,s) = ce^{-\lambda t} + e^{-\lambda t}(0) = ce^{-\lambda t}$$
$$q_0(s,s) = 1 = ce^{-\lambda s}$$

Therefore $c = \frac{1}{e^{-\lambda s}} = e^{\lambda s}$ and $q_0(t,s) = e^{\lambda s}e^{-\lambda t} = e^{-\lambda(t-s)}$

$$q_1(t,s) = ce^{-\lambda t} + e^{-\lambda t}\int e^{\lambda t}\lambda e^{-\lambda(t-s)}dt = ce^{-\lambda t} + e^{-\lambda t}e^{\lambda s}\lambda\int dt$$
$$= ce^{-\lambda t} + \lambda e^{-\lambda(t-s)} \cdot t = ce^{-\lambda t} + \lambda t e^{-\lambda(t-s)}$$
$$q_1(s,s) = 0 = ce^{-\lambda s} + \lambda s \Rightarrow ce^{-\lambda s} = -\lambda s \Rightarrow c = -\lambda s e^{\lambda s}$$

Therefore

$$q_1(t,s) = -\lambda s e^{\lambda s}e^{-\lambda t} + \lambda t e^{-\lambda(t-s)} = -\lambda s e^{-\lambda(t-s)} + \lambda t e^{-\lambda(t-s)} = \lambda(t-s)e^{-\lambda(t-s)}$$

etc...

Therefore, $q_n(t,s) = p_n(t-s)$. Similarly for $q_n(t+h, s+h)$.

1.17 Let $P_n(t) \equiv CDF$ of the arrival counting process.

Then,

$$\begin{aligned} P_n(t) &= \Pr\{(\text{sum of } n+1 \text{Erlang interarrival times}) \geq t\} \\ &= \int_t^\infty \frac{k\lambda(k\lambda x)^{(n+1)k-1}}{[(n+1)k-1]!} e^{-k\lambda x} dx \end{aligned}$$

since the sum of IID Erlang random variables is also an Erlang.

Let $u = x - t$,

$$\begin{aligned} P_n(t) &= \int_0^\infty \frac{(k\lambda)^{(n+1)k}(u+t)^{(n+1)k-1}}{[(n+1)k-1]!} e^{-k\lambda u} e^{-k\lambda t} du \\ &= \int_0^\infty \frac{(k\lambda)^{(n+1)k} e^{-k\lambda u} e^{-k\lambda t}}{[(n+1)k-1]!} \sum_{i=0}^{(n+1)k-1} \frac{u^{(n+1)k-1-i} t^i}{[(n+1)k-1-i]!} \cdot \frac{[(n+1)k-1]!}{i!} du \\ &= \sum_{i=0}^{(n+1)k-1} \frac{(k\lambda)^{(n+1)k} t^i e^{-k\lambda t}}{[(n+1)k-1-i]! i!} \cdot \int_0^\infty e^{-k\lambda u} u^{(n+1)k-1-i} du \\ &= \sum_{i=0}^{(n+1)k-1} \frac{(k\lambda)^{(n+1)k} t^i e^{-k\lambda t}}{[(n+1)k-1-i]! i!} \cdot \frac{[(n+1)k-1-i]!}{(k\lambda)^{(n+1)k-1}} = \sum_{i=0}^{(n+1)k-1} \frac{(k\lambda t)^i}{i!} e^{-k\lambda t} \end{aligned}$$

The probability function of the counting process is thus,

$$\begin{aligned} p_n(t) = P_n(t) - P_{n-1}(t) &= \sum_{i=0}^{(n+1)k-1} \frac{(k\lambda t)^i}{i!} e^{-k\lambda t} - \sum_{i=0}^{nk-1} \frac{(k\lambda t)^i}{i!} e^{-k\lambda t} \\ &= \sum_{i=nk}^{(n+1)k-1} \frac{(k\lambda t)^i}{i!} e^{-k\lambda t} \end{aligned}$$

1.18 First, assume that n is even. Then,

$$\begin{aligned} p_n(t) &= \Pr\{N(t) = n\} \\ &= \Pr\{n \text{ singles}\} + \Pr\{(n-2) \text{ singles and 1 double}\} \\ &\quad + \Pr\{(n-4) \text{ singles and 2 doubles}\} + \cdots + \Pr\{n/2 \text{ doubles}\}. \end{aligned}$$

Then,

$$p_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}p^n + \binom{n-1}{1} e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}p^{n-2}(1-p)$$
$$+ \binom{n-2}{2} e^{-\lambda t}\frac{(\lambda t)^{n-2}}{(n-2)!}p^{n-4}(1-p)^2 + \cdots$$
$$+ \binom{n-n/2}{n/2} e^{-\lambda t}\frac{(\lambda t)^{n-n/2}}{(n-n/2)!}p^{n-2(n/2)}(1-p)^{n/2}.$$

So,

$$\begin{aligned} p_n(t) &= e^{-\lambda t}\left\{\frac{(\lambda t)^n}{n!}p^n + \frac{(\lambda t)^{n-1}}{1!(n-2)!}p^{n-2}(1-p)\right. \\ &\qquad \left. + \frac{(\lambda t)^{n-2}}{2!(n-4)!}p^{n-4}(1-p)^2 + \cdots + \frac{(\lambda t)^{n/2}}{(n/2)!}(1-p)^{n/2}\right\} \\ &= e^{-\lambda t}\sum_{k=0}^{n/2}\frac{(\lambda t)^{n-k}}{k!(n-2k)!}p^{n-2k}(1-p)^k. \end{aligned}$$

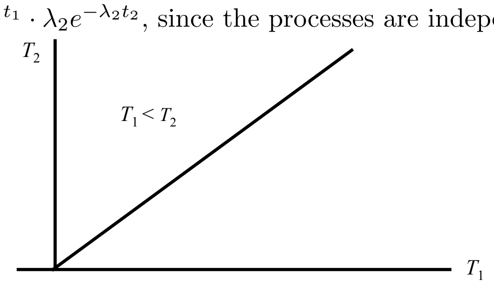
Similarly, if n is odd,

$$\begin{aligned} p_n(t) &= \Pr\{N(t)=n\} \\ &= \Pr\{n \text{ singles}\} + \Pr\{(n-2) \text{ singles and 1 double}\} \\ &\quad + \Pr\{(n-4) \text{ singles and 2 doubles}\} \\ &\quad + \cdots + \Pr\{1 \text{ single and } (n-1)/2 \text{ doubles}\}. \end{aligned}$$

Proceeding in the same manner gives

$$p_n(t) = e^{-\lambda t}\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(\lambda t)^{n-k}}{k!(n-2k)!}p^{n-2k}(1-p)^k.$$

1.19 **(a)** Denote respective first recurrence times as T_1 and T_2. The joint PDF is $f(t_1,t_2) = \lambda_1 e^{-\lambda_1 t_1}\cdot\lambda_2 e^{-\lambda_2 t_2}$, since the processes are independent.



$$P(T_1 < T_2) = \int_0^\infty \int_0^{t_2} f(t_1, t_2) dt_1, dt_2 = \int_0^\infty \int_0^{t_2} \lambda_1 e^{-\lambda_1 t_1} \lambda_2 e^{-\lambda_2 t_2} dt_1 dt_2$$
$$= \int_0^\infty (1 - e^{-\lambda_1 t_2}) \lambda_2 e^{-\lambda_2 t_2} dt_2 = 1 - \frac{\lambda_2}{\lambda_1 + \lambda_2} = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

(b) First n services will take on the average $n/m\mu$ while the remaining m will require $\sum_{i=1}^{m} (1/i\mu)$.

1.20 Equation (1.25): $p_{jk}(u, s) = \sum_i p_{ji}(u, t) p_{ik}(t, s)$

Equation (1.27b): $p_{jk}(t, t + \Delta t) = \begin{cases} q_{jk}(t)\Delta t + o(\Delta t) & j \neq k \\ 1 - q_j(t)\Delta t + o(\Delta t) & j = k \end{cases}$

Let $s = t + \Delta t$ in (1.26)

$$p_{jk}(u, t + \Delta t) = \sum_i p_{ji}(u, t) p_{ik}(t, t + \Delta t)$$
$$= p_{jk}(u, t) p_{kk}(t, t + \Delta t) + \sum_{i \neq k} p_{ji}(u, t) p_{ik}(t, t + \Delta t)$$

Introducing (1.27b)

$$p_{jk}(u, t + \Delta t) = p_{jk}(u, t)[1 - q_k(t)\Delta t + o(\Delta t)]$$
$$+ \sum_{i \neq k} p_{ji}(u, t)[q_{ik}(t)\Delta t + o(\Delta t)]$$

Rewriting this,

$$p_{jk}(u, t + \Delta t) - p_{jk}(u, t) = [-q_k(t)\Delta t + o(\Delta t)] p_{jk}(u, t)$$
$$+ \sum_{i \neq k} p_{ji}(u, t)[q_{ik}(t)\Delta t + o(\Delta t)]$$

Dividing by Δt and taking the limit as $\Delta t \to 0$,

$$\frac{\partial p_{jk}(u, t)}{\partial t} = -q_k(t) p_{jk}(u, t) + \sum_{i \neq k} p_{ji}(u, t) q_{ik}(t)$$

which is the Kolmogorov forward equation (1.28a) Next, let $u = t - \Delta t$ in (1.26) and then introduce (1.27b), [or let $s = t$; $t = u + \Delta u$]

$$\begin{aligned} p_{jk}(t-\Delta t, s) &= \sum_i p_{ji}(t-\Delta t, t) p_{ik}(t,s) \\ &= p_{jj}(t-\Delta t, t) p_{jk}(t,s) + \sum_{i\neq j} p_{ji}(t-\Delta t, t) p_{ik}(t,s) \\ &= [1 - q_j(t)\Delta t + o(\Delta t)] p_{jk}(t,s) + \sum_{i\neq j} [q_{ji}(t)\Delta t + o(\Delta t)] p_{ik}(t,s) \end{aligned}$$

Rewriting and dividing by Δt,

$$\frac{p_{jk}(t-\Delta t, s) p_{jk}(t,s)}{\Delta t} = -q_j(t) p_{jk}(t,s) + \frac{o(\Delta t)}{\Delta t} + \sum_{i\neq j} \left[q_{ji}(t) + \frac{o(\Delta t)}{\Delta t} \right] p_{ik}(t,s)$$

Taking the limit as $\Delta t \to 0$,

$$-\frac{\partial p_{jk}(t,s)}{\partial t} = -q_j(t) p_{jk}(t,s) + \sum_{i\neq j} q_{ji}(t) p_{ik}(t,s)$$

Let $t = u$ and $s = t$ to get the Kolmogorov backward equation (1.28b),

$$\frac{\partial p_{jk}(u,t)}{\partial u} = q_i(u) p_{jk}(u,t) - \sum_{i\neq j} q_{ji}(u) p_{ik}(u,t)$$

1.21 Let

$$\boldsymbol{Q} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & \cdots \\ 0 & -\lambda & \lambda & 0 & 0 & \cdots \\ 0 & 0 & -\lambda & \lambda & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

Then $\boldsymbol{p}'(t) = \boldsymbol{p}(t)\boldsymbol{Q}$ gives

$$(p_0'(t), p_1'(t), \ldots) = (p_0(t), p_1(t), \ldots) \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & \cdots \\ 0 & -\lambda & \lambda & 0 & 0 & \cdots \\ 0 & 0 & -\lambda & \lambda & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix},$$

which yields

$$\begin{aligned} p_0'(t) &= -\lambda p_0(t), \\ p_1'(t) &= \lambda p_0(t) - \lambda p_1(t), \\ p_2'(t) &= \lambda p_1(t) - \lambda p_2(t), \\ &\vdots \\ p_n'(t) &= \lambda p_{n-1}(t) - \lambda p_n(t). \end{aligned}$$

1.22 (a)

MARKOV CHAIN

To start a new problem, enter number of states and press button.

Number of States: 2

Enter transition probabilities, then press "Solve" button.

Solve

Solution		**Enter P Matrix below**	
p0	0.5	0.	1.
p1	0.5	1.	0.

MARKOV CHAIN

To start a new problem, enter number of states and press button.

Number of States: 2

Enter transition probabilities, then press "Solve" button.

Solve

Solution		**Enter P Matrix below**	
p0	0.5	0.5	0.5
p1	0.5	0.5	0.5

MARKOV CHAIN

To start a new problem, enter number of states and press button.

Number of States: 2

Enter transition probabilities, then press "Solve" button.

Solve

Solution		**Enter P Matrix below**	
p0	0.5	0.333333	0.666667
p1	0.5	0.666667	0.333333

(b)

MARKOV CHAIN

To start a new problem, enter number of states and press button.

Number of States: 4

Enter transition probabilities, then press "Solve" button.

Solve

Solution		**Enter P Matrix below**			
p0	0.025615	0.25	0.2	0.12	0.43
p1	0.076844	0.25	0.2	0.12	0.43
p2	0.22541	0.	0.25	0.2	0.55
p3	0.672131	0.	0.	0.25	0.75

1.23 $i \geq 1$:

$$
\begin{aligned}
&\Pr\{\text{goes to state } i+1 \mid \text{in state } i \text{ and a transition occurs}\}\\
&\quad = \frac{\Pr\{\text{goes to state } i+1 \text{ and a trans. occurs} \mid \text{in state } i\}}{\Pr\{\text{trans. occurs} \mid \text{in state } i\}}\\
&\quad = \lim_{\Delta t \to 0} \left\{ \frac{\lambda_i \Delta t + o(\Delta t)}{1 - [1 - \lambda_i \Delta t - \mu_i \Delta t + o(\Delta t)]} \right\} = \lim_{\Delta t \to 0} \left\{ \frac{\lambda_i + o(\Delta t)/\Delta t}{\lambda_i + \mu_i + o(\Delta t)/\Delta t} \right\} = \frac{\lambda_i}{\lambda_i + \mu_i}
\end{aligned}
$$

Similarly, $\Pr\{\text{goes to state } i-1 \mid \text{in state } i \text{ and a transition occurs}\} = \cdots$

$$
= \frac{\mu_i}{\lambda_i + \mu_i}.
$$

Or, using the results of Problem 1.19a, two independent Poisson processes with parameters λ_i and μ_i, respectively, yield

$$
\begin{aligned}
P\{\text{arrival before a departure}\} &= \frac{\lambda_i}{\lambda_i + \mu_i},\\
P\{\text{departure before an arrival}\} &= \frac{\mu_i}{\lambda + \mu_i}.
\end{aligned}
$$

1.24

$$
\begin{aligned}
P(T_{(1)} \leq t) &= 1 - (1-t)^n\\
P(nT_{(1)} \leq t) &= P\left(T_{(1)} \leq \frac{t}{n}\right) = 1 - \left(1 - \frac{t}{n}\right)^n.
\end{aligned}
$$

Since

$$
\lim_{n \to \infty} \left(1 - \frac{t}{n}\right)^n = e^{-t},
$$

then $\lim_{n \to \infty} P(nT_{(1)} \leq t) = 1 - e^{-t}$, the exponential CDF.

1.25

$$
\begin{aligned}
&\lambda_{\text{eff}} = \lambda(1 - p_K) = .9; W = L/\lambda_{\text{eff}} = 5/.9 = 50/9;\\
&W_q = W - 1/\mu = 50/9 - 1 = 41/9;\\
&\rho_{\text{eff}} = \lambda_{\text{eff}}/\mu = .9 \text{ and } p_0 = 1 - \rho_{\text{eff}} = .1.
\end{aligned}
$$

1.26 **(a)** $CV = 1.6/2.25 = .71111 = 1/\sqrt{k}$. Therefore, $k = (1/.7111)^2 = 1.971$(approximately 2). Now $\beta = 1.6/\sqrt{2} = 1.13$. Using the QtsPlus Basic Erlang Probability Calculator we get:

ERLANG(k) PROBABILITY CALCULATIONS

Input Parameters:

Erlang scale parameter (β)	1.13
Erlang shape parameter (k)	2

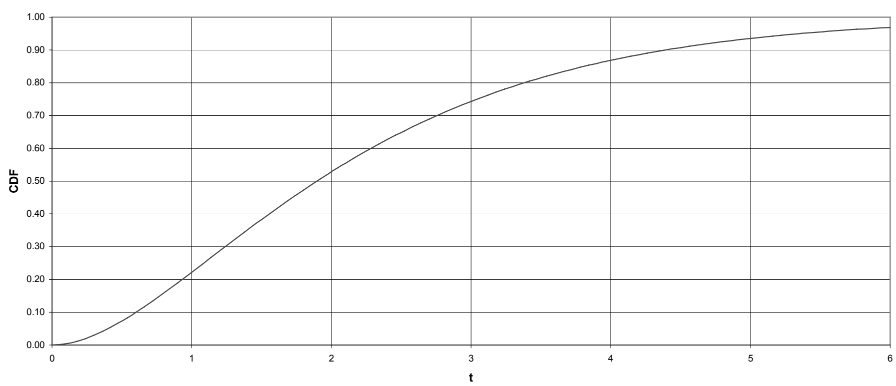
Plot Parameters:

Maximum print/plot value	6

Results:

Distribution mean	2.26
Distribution variance	2.5538
Coefficient of Variation	0.707107

Erlang(2) Distribution Function



So $\Pr(\text{Service} \geq 5 \text{ min}) \approx 1 - 0.93 = .07$.

(b) Using the QtsPlus Mixed Exponential Probability Computation Module (here we have a pure exponential or a mixture of 1):

MIXED EXPONENTIAL PROBABILITY COMPUTATIONS

Note that we allow both regular exponential mixtures, which are convex combinations of negative exponential functions, as well as generalized mixtures where the proportionality constants may be negative although still summing to 1.

Clear Old Data

Input Parameters:

p_i	θ_i
1.	1.06

Plot Parameters:

Maximum print/plot value	10

Results:

Mean value	1.06
Second moment about zero	2.2472
Variance	1.1236
Coefficient of variation	1